New n-mode squeezing operator and squeezed states with standard squeezing *

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Abstract

We find that the exponential operator $V \equiv \exp\left[\mathtt{i}\lambda\left(Q_1P_2 + Q_2P_3 + \cdots + Q_{n-1}P_n + Q_nP_1\right)\right]$, Q_i, P_i are respectively the coordinate and momentum operators, is an n-mode squeezing operator which engenders standard squeezing. By virtue of the technique of integration within an ordered product of operators we derive V's normally ordered expansion and obtain the n-mode squeezed vacuum states, its Wigner function is calculated by using the Weyl ordering invariance under similar transformations.

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1 Introduction

Quantum entanglement is a weird, remarkable feature of quantum mechanics though it implies intricacy. In recent years, various entangled states have brought considerable attention and interests of physicists because of their potential uses in quantum communication [1, 2]. Among them the two-mode squeezed state exhibits quantum entanglement between the idle-mode and the signal-mode in a frequency domain manifestly, and is a typical entangled state of continuous variable. Theoretically, the two-mode squeezed state is constructed by the two-mode squeezing operator $S = \exp[\lambda(a_1a_2 - a_1^\dagger a_2^\dagger)]$ [3, 4, 5] acting on the two-mode vacuum state $|00\rangle$,

$$S\left|00\right\rangle = \mathrm{sech}\lambda \exp\left[-a_1^\dagger a_2^\dagger \tanh\lambda\right]\left|00\right\rangle, \tag{1}$$

where λ is a squeezing parameter, the disentangling of S can be obtained by using SU(1,1) Lie algebra, $[a_1a_2, a_1^\dagger a_2^\dagger] = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$, or by using the entangled state representation $|\eta = \eta_1 + i\eta_2\rangle$ [6, 7]

$$\left|\eta\right\rangle = \exp\left[-\frac{1}{2}\left|\eta\right|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]\left|00\right\rangle, \tag{2}$$

$|\eta\rangle$ is the common eigenvector of two particles' relative position $(Q_1 - Q_2)$ and the tota momentum $(P_1 + P_2)$, obeys the eigenvector equation, $(Q_1 - Q_2)\left|\eta\right\rangle = \sqrt{2}\eta_1\left|\eta\right\rangle$, $(P_1 + P_2) = \left|\eta\right\rangle = \sqrt{2}\eta_2\left|\eta\right\rangle$, and the orthonormal-complete relation

$$\int \frac{d^2\eta}{\pi}\left|\eta\right\rangle\left\langle\eta\right| = 1, \ \ \left\langle\eta'\right|\left.\eta\right\rangle = \pi\delta\left(\eta - \eta'\right)\left(\eta^* - \eta'^*\right), \tag{3}$$

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because the two-mode squeezing operator has its natural representation in $\langle\eta|$ basis

$$S=\exp\left[\lambda\left(a_1a_2-a_1^\dagger a_2^\dagger\right)\right]=\int\frac{d^2\eta}{\pi\mu}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|\,,\ S\,|\eta\rangle=\frac{1}{\mu}\left|\frac{\eta}{\mu}\right\rangle,\ \mu=e^{\lambda}, \tag{4}$$

The proof of Eq.(4) is proceeded by virtue of the technique of integration within an ordered product (IWOP) of operators [8, 9, 10]

$$\begin{aligned}\int\frac{d^2\eta}{\pi\mu}\,|\eta/\mu\rangle\langle\eta| &= \int\frac{d^2\eta}{\pi\mu}\colon\exp\left\{-\frac{\mu^2+1}{2\mu^2}|\eta|^2+\eta\left(\frac{a_1^\dagger}{\mu}-a_2\right)\right.\\ &\quad\left.+\eta^*\left(a_1-\frac{a_2^\dagger}{\mu}\right)+a_1^\dagger a_2^\dagger+a_1a_2-a_1^\dagger a_1-a_2^\dagger a_2\right\}\colon\\ &= \frac{2\mu}{1+\mu^2}\colon\exp\left\{\frac{\mu^2}{1+\mu^2}\left(\frac{a_1^\dagger}{\mu}-a_2\right)\left(a_1-\frac{a_2^\dagger}{\mu}\right)-\left(a_1-a_2^\dagger\right)\left(a_1^\dagger-a_2\right)\right\}\colon\\ &= e^{-a_1^\dagger a_2^\dagger\tanh\lambda}e^{(a_1^\dagger a_1+a_2^\dagger a_2+1)\ln\mathrm{sech}\lambda}e^{a_1a_2\tanh\lambda}\equiv S,\end{aligned} \tag{5}$$

Eq. (4) confirms that the two-mode squeezed state itself is an entangled state which entangles the idle mode and signal mode as an outcome of a parametric-down conversion process [11]. The $|\eta\rangle$ state was constructed in Ref. [6, 7] according to the idea of Einstein, Podolsky and Rosen in their argument that quantum mechanics is incomplete [12].

Using the relation between bosonic operators and the coordinate Q_i, momentum P_i, $Q_i=(a_i+a_i^\dagger)/\sqrt{2}$, $P_i=(a_i-a_i^\dagger)/(\sqrt{2}\mathtt{i})$, and introducing the two-mode quadrature operators of light field as in Ref. [4], $x_1=(Q_1+Q_2)/2, x_2=(P_1+P_2)/2$, the variances of x_1 and x_2 in the state $S\,|00\rangle$ are in the standard form

$$\langle 00|\,S^\dagger x_2^2S\,|00\rangle=\frac{1}{4}e^{-2\lambda},\quad \langle 00|\,S^\dagger x_1^2S\,|00\rangle=\frac{1}{4}e^{2\lambda}, \tag{6}$$

thus we get the standard squeezing for the two quadrature: $x_1\to\frac{1}{2}e^{\lambda}x_1, x_2\to\frac{1}{2}e^{-\lambda}x_2$. On the other hand, the two-mode squeezing operator can also be recast into the form $S=\exp\left[\mathtt{i}\lambda\left(Q_1P_2+Q_2P_1\right)\right].$ Then an interesting question naturally rises: what is the property of the n-mode operator

$$V\equiv\exp\left[\mathtt{i}\lambda\left(Q_1P_2+Q_2P_3+\cdots+Q_{n-1}P_n+Q_nP_1\right)\right], \tag{7}$$

and is it a squeezing operator which can engenders the standard squeezing for n-mode quadratures? What is the normally ordered expansion of V and what is the state $V\,|\mathbf{0}\rangle$ ($|\mathbf{0}\rangle$ is the n-mode vacuum state)? In this work we shall study V in detail. But how to disentangling the exponential of V? Since all terms of the set Q_iP_{i+1} $(i=1\cdots n)$ do not make up a closed Lie algebra, the problem of what is V's the normally ordered form seems difficult. Thus we appeal to the IWOP technique to solve this problem. Our work is arranged in this way: firstly we use the IWOP technique to derive the normally ordered expansion of V and obtain the explicit form of $V\,|\mathbf{0}\rangle$; then we examine the variances of the n-mode quadrature operators in the state $V\,|\mathbf{0}\rangle$, we find that V just causes standard squeezing. Thus V is a squeezing operator. The Wigner function of $V\,|\mathbf{0}\rangle$ is calculated by using the Weyl ordering invariance under similar transformations. Some examples are discussed in the last section.

2 The normal product form of V

In order to disentangle operator V, let A be

$$A=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\ddots&\ddots&0\\0&0&\cdots&\ddots&1\\1&0&\cdots&\cdots&0\end{pmatrix}, \tag{8}$$

then V in (7) is compactly expressed as

$$V=\exp\left[\mathtt{i}\lambda\sum_{i,j=1}^{n}Q_iA_{ij}P_j\right]. \tag{9}$$

Using the Baker-Hausdorff formula, $e^{A}Be^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\cdots$,we have (here and henceforth the repeated indices represent the Einstein summation notation)

$$\begin{aligned}V^{-1}Q_kV &= Q_k-\lambda Q_iA_{ik}+\frac{1}{2!}\mathtt{i}\lambda^2\left[Q_iA_{ij}P_j,Q_lA_{lk}\right]+... \\ &= Q_i(e^{-\lambda A})_{ik}=(e^{-\lambda\tilde{A}})_{ki}Q_i, \qquad (10)\\ V^{-1}P_kV &= P_k+\lambda A_{ki}P_i+\frac{1}{2!}\mathtt{i}\lambda^2\left[A_{ki}P_j,Q_lA_{lm}P_m\right]+... \\ &= (e^{\lambda A})_{ki}P_i, \qquad (11)\end{aligned}$$

From Eq.(10) we see that when V acts on the n-mode coordinate eigenstate $|\vec{q}\rangle$, where $\widetilde{\vec{q}}=(q_1,q_2,\cdots,q_n)$, it squeezes $|\vec{q}\rangle$ in the way of

$$V|\vec{q}\rangle=|\Lambda|^{1/2}|\Lambda\vec{q}\rangle,\ \Lambda=e^{-\lambda\tilde{A}},\ \ |\Lambda|\equiv\det\Lambda. \tag{12}$$

Thus V has the representation on the coordinate $\langle\vec{q}|$ basis

$$V=\int d^nqV|\vec{q}\rangle\langle\vec{q}|=|\Lambda|^{1/2}\int d^nq\,|\Lambda\vec{q}\rangle\langle\vec{q}|,\quad V^{\dagger}=V^{-1}, \tag{13}$$

since $\int d^nq\,|\vec{q}\rangle\langle\vec{q}|=1$. Using the expression of eigenstate $|\vec{q}\rangle$ in Fock space

$$\begin{gathered}|\vec{q}\rangle=\pi^{-n/4}\colon\exp[-\tfrac{1}{2}\widetilde{\vec{q}}\vec{q}+\sqrt{2}\widetilde{\vec{q}}a^{\dagger}-\tfrac{1}{2}\tilde{a}^{\dagger}a^{\dagger}]\,|\mathbf{0}\rangle, \qquad (14)\\ \tilde{a}^{\dagger}=(a_1^{\dagger},a_2^{\dagger},\cdots,a_n^{\dagger}),\end{gathered}$$

and $|\mathbf{0}\rangle\langle\mathbf{0}|=\colon\exp[-\tilde{a}^{\dagger}a^{\dagger}]\colon$, we can put V into the normal ordering form ,

$$\begin{aligned}V &= \pi^{-n/2}|\Lambda|^{1/2}\int d^nq\colon\exp[-\frac{1}{2}\widetilde{\vec{q}}(1+\widetilde{\Lambda}\Lambda)\vec{q}+\sqrt{2}\widetilde{\vec{q}}(\widetilde{\Lambda}a^{\dagger}+a) \\ &\quad -\frac{1}{2}(\tilde{a}a+\tilde{a}^{\dagger}a^{\dagger})-\tilde{a}^{\dagger}a]\colon. \qquad (15)\end{aligned}$$

To compute the integration in Eq.(15) by virtue of the IWOP technique, we use the mathematical formula

$$\int d^nx\exp[-\tilde{x}Fx+\tilde{x}v]=\pi^{n/2}(\det F)^{-1/2}\exp\left[\frac{1}{4}\tilde{v}F^{-1}v\right], \tag{16}$$

then we derive

$$\begin{aligned}V &= \left(\frac{\det\Lambda}{\det N}\right)^{1/2}\exp\left[\frac{1}{2}\tilde{a}^{\dagger}\left(\Lambda N^{-1}\widetilde{\Lambda}-I\right)a^{\dagger}\right] \\ &\quad\times\colon\exp\left[\tilde{a}^{\dagger}\left(\Lambda N^{-1}-I\right)a\right]\colon\exp\left[\frac{1}{2}\tilde{a}\left(N^{-1}-I\right)a\right], \qquad (17)\end{aligned}$$

where $N=(1+\widetilde{\Lambda}\Lambda)/2$. Eq.(17) is just the normal product form of V.

3 The squeezing property of $V|\mathbf{0}\rangle$

Operating V on the n-mode vacuum state $|\mathbf{0}\rangle$, we obtain the squeezed vacuum state

$$V|\mathbf{0}\rangle=\left(\frac{\det\Lambda}{\det N}\right)^{1/2}\exp\left[\frac{1}{2}\tilde{a}^{\dagger}\left(\Lambda N^{-1}\widetilde{\Lambda}-I\right)a^{\dagger}\right]|\mathbf{0}\rangle. \tag{18}$$

Now we evaluate the variances of the n-mode quadratures. The quadratures in the n-mode case are defined as

$$X_1 = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} Q_i, \;\; X_2 = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} P_i, \tag{19}$$

obeying $[X_1, X_2] = \frac{\mathtt{i}}{2}$. Their variances are $(\Delta X_i)^2 = \langle X_i^2 \rangle - \langle X_i \rangle^2$, $i = 1, 2$. Noting the expectation values of X_1 and X_2 in the state $V\,|\mathbf{0}\rangle$, $\langle X_1 \rangle = \langle X_2 \rangle = 0$, and using Eqs. (10) and (11) we see that the variances are

$$\begin{aligned}
(\triangle X_1)^2 &= \langle \mathbf{0}|\, V^{-1} X_1^2 V\, |\mathbf{0}\rangle = \frac{1}{2n} \langle \mathbf{0}|\, V^{-1} \sum_{i=1}^{n} Q_i \sum_{j=1}^{n} Q_j V\, |\mathbf{0}\rangle \\
&= \frac{1}{2n} \langle \mathbf{0}| \sum_{i=1}^{n} Q_k (e^{-\lambda A})_{ki} \sum_{j=1}^{n} (e^{-\lambda \tilde{A}})_{jl} Q_l\, |\mathbf{0}\rangle \\
&= \frac{1}{2n} \sum_{i,j}^{n} (e^{-\lambda A})_{ki} (e^{-\lambda \tilde{A}})_{jl} \langle \mathbf{0}|\, Q_k Q_l\, |\mathbf{0}\rangle \\
&= \frac{1}{4n} \sum_{i,j}^{n} (e^{-\lambda A})_{ki} (e^{-\lambda \tilde{A}})_{jl} \langle \mathbf{0}|\, a_k a_l^\dagger\, |\mathbf{0}\rangle \\
&= \frac{1}{4n} \sum_{i,j}^{n} (e^{-\lambda A})_{ki} (e^{-\lambda \tilde{A}})_{jl} \delta_{kl} = \frac{1}{4n} \sum_{i,j}^{n} (\widetilde{\Lambda}\Lambda)_{ij},
\end{aligned} \tag{20}$$

similarly we have

$$(\triangle X_2)^2 = \langle \mathbf{0}|\, V^{-1} X_2^2 V\, |\mathbf{0}\rangle = \frac{1}{4n} \sum_{i,j}^{n} (\widetilde{\Lambda}\Lambda)^{-1}_{ij}, \tag{21}$$

Eqs. (20) -(21) are the quadrature variance formula in the transformed vacuum state acted by the operator $\exp[\mathtt{i}\lambda \sum\limits_{i,j=1}^{n} Q_i A_{ij} P_j]$. By observing that A in (8) is a cyclic matrix, we see

$$\sum_{i,j}^{n} \left[(A + \tilde{A})^l \right]_{i\,j} = 2^l n, \tag{22}$$

then using $A\tilde{A} = \tilde{A}A$, so $\widetilde{\Lambda}\Lambda = e^{-\lambda(A+\tilde{A})}$, a symmetric matrix, we have

$$\sum_{i,j=1}^{n} (\widetilde{\Lambda}\Lambda)_{i\,j} = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \sum_{i,j}^{n} \left[(A + \tilde{A})^l \right]_{i\,j} = n \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} 2^l = n e^{-2\lambda}, \tag{23}$$

and

$$\sum_{i,j=1}^{n} (\widetilde{\Lambda}\Lambda)^{-1}_{i\,j} = n e^{2\lambda}. \tag{24}$$

it then follows

$$(\triangle X_1)^2 = \frac{1}{4n} \sum_{i,j}^{n} (\widetilde{\Lambda}\Lambda)_{ij} = \frac{e^{-2\lambda}}{4}, \tag{25}$$

$$(\triangle X_2)^2 = \frac{1}{4n} \sum_{i,j}^{n} (\widetilde{\Lambda}\Lambda)^{-1}_{ij} = \frac{e^{2\lambda}}{4}. \tag{26}$$

This leads to $\triangle X_1 \cdot \triangle X_2 = \frac{1}{4}$, which shows that V is a correct n-mode squeezing operator for the n-mode quadratures in Eq.(19) and produces the standard squeezing similar to Eq. (6).

4 The Wigner function of $V\left|\mathbf{0}\right\rangle$

Wigner distribution functions [13, 14, 15] of quantum states are widely studied in quantum statistics and quantum optics. Now we derive the expression of the Wigner function of $V\left|\mathbf{0}\right\rangle$. Here we take a new method to do it. Recalling that in Ref.[16, 17, 18] we have introduced the Weyl ordering form of single-mode Wigner operator $\Delta\left(q,p\right)$,

$$\Delta_1\left(q_1,p_1\right)=\vdots\,\delta\left(q_1-Q_1\right)\delta\left(p_1-P_1\right)\vdots, \tag{27}$$

its normal ordering form is

$$\Delta_1\left(q_1,p_1\right)=\frac{1}{\pi}\colon\exp\left[-\left(q_1-Q_1\right)^2-\left(p_1-P_1\right)^2\right]\colon \tag{28}$$

where the symbols $:\ :$ and $\vdots\ \vdots$ denote the normal ordering and the Weyl ordering, respectively. Note that the order of Bose operators a_1 and $a_1^\dagger$ within a normally ordered product and a Weyl ordered product can be permuted. That is to say, even though $[a_1,a_1^\dagger]=1$, we can have $:a_1a_1^\dagger:\ =:a_1^\dagger a_1:$ and $\vdots\, a_1a_1^\dagger\,\vdots=\vdots\, a_1^\dagger a_1\,\vdots$. The Weyl ordering has a remarkable property, i.e., the order-invariance of Weyl ordered operators under similar transformations [16, 17, 18], which means

$$U\vdots\left(\circ\circ\circ\right)\vdots U^{-1}=\vdots\, U\left(\circ\circ\circ\right)U^{-1}\vdots, \tag{29}$$

as if the "fence" $\vdots\ \vdots$ did not exist.

For n-mode case, the Weyl ordering form of the Wigner operator is

$$\Delta_n\left(\vec{q},\vec{p}\right)=\vdots\,\delta\left(\vec{q}-\vec{Q}\right)\delta\left(\vec{p}-\vec{P}\right)\vdots, \tag{30}$$

where $\widetilde{\vec{Q}}=\left(Q_1,Q_2,\cdots,Q_n\right)$ and $\widetilde{\vec{P}}=\left(P_1,P_2,\cdots,P_n\right)$. Then according to the Weyl ordering invariance under similar transformations and Eqs.(10) and (11) we have

$$\begin{aligned}
V^{-1}\Delta_n\left(\vec{q},\vec{p}\right)V &= V^{-1}\vdots\,\delta\left(\vec{q}-\vec{Q}\right)\delta\left(\vec{p}-\vec{P}\right)\vdots\, V\\
&= \vdots\,\delta\left(q_k-(e^{-\lambda\tilde{A}})_{ki}Q_i\right)\delta\left(p_k-(e^{rA})_{ki}P_i\right)\vdots\\
&= \vdots\,\delta\left(e^{r\tilde{A}}\vec{q}-\vec{Q}\right)\delta\left(e^{-rA}\vec{p}-\vec{P}\right)\vdots\\
&= \Delta\left(e^{r\tilde{A}}\vec{q},e^{-rA}\vec{p}\right),
\end{aligned} \tag{31}$$

Thus using Eqs.(27) and (31) the Wigner function of $V\left|\mathbf{0}\right\rangle$ is

$$\begin{aligned}
&\left\langle\mathbf{0}\right|V^{-1}\Delta_n\left(\vec{q},\vec{p}\right)V\left|\mathbf{0}\right\rangle\\
&= \frac{1}{\pi^n}\left\langle\mathbf{0}\right|\colon\exp[-(e^{r\tilde{A}}\vec{q}-\vec{Q})^2-(e^{-rA}\vec{p}-\vec{P})^2]\colon\left|\mathbf{0}\right\rangle\\
&= \frac{1}{\pi^n}\exp[-(e^{r\tilde{A}}\vec{q})^2-\left(e^{-rA}\vec{p}\right)^2]\\
&= \frac{1}{\pi^n}\exp\left[-\widetilde{\vec{q}}e^{rA}e^{r\tilde{A}}\vec{q}-\widetilde{\vec{p}}e^{-r\tilde{A}}e^{-rA}\vec{p}\right]\\
&= \frac{1}{\pi^n}\exp\left[-\widetilde{\vec{q}}\left(\Lambda\widetilde{\Lambda}\right)^{-1}\vec{q}-\widetilde{\vec{p}}\Lambda\widetilde{\Lambda}\vec{p}\right],
\end{aligned} \tag{32}$$

From Eq.(32) we see that once the explicit expression of $\Lambda\tilde{\Lambda}=\exp[-\lambda(A+\tilde{A})]$ is deduced, the Wigner function of $V\left|\mathbf{0}\right\rangle$ can be calculated.

5 Some examples of calculating the Wigner function

Taking $n = 2$ as an example, $V_{n=2}$ is the usual two-mode squeezing operator. The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it then follows that

$$\Lambda\tilde{\Lambda} = e^{-\lambda(\tilde{A}+A)} = \begin{pmatrix} \cosh 2\lambda & -\sinh 2\lambda \\ -\sinh 2\lambda & \cosh 2\lambda \end{pmatrix}, \tag{33}$$

and

$$\left(\Lambda\tilde{\Lambda}\right)^{-1} = \begin{pmatrix} \cosh 2\lambda & \sinh 2\lambda \\ \sinh 2\lambda & \cosh 2\lambda \end{pmatrix}. \tag{34}$$

Substituting Eqs.(33) and (34) into Eq.(32), we have

$$\langle 00| V^{-1}\Delta_2(\vec{q},\vec{p}) V |00\rangle = \frac{1}{\pi^2}\exp\left[-2\left(\alpha_1^*\alpha_2^* + \alpha_1\alpha_2\right)\sinh 2\lambda - 2\left(|\alpha_1|^2 + |\alpha_2|^2\right)\cosh 2\lambda\right], \tag{35}$$

where $\alpha_i = \frac{1}{\sqrt{2}}(q_i + \mathtt{i}p_i), (i = 1, 2.)$. Eq.(35) is just the Wigner function of the usual two-mode squeezing vacuum state. For $n = 3$, we have

$$\Lambda\tilde{\Lambda} = \begin{pmatrix} u & v & v \\ v & u & v \\ v & v & u \end{pmatrix}, \; u = \frac{2}{3}e^{\lambda} + \frac{1}{3}e^{-2\lambda}, \; v = \frac{1}{3}\left(e^{-2\lambda} - e^{\lambda}\right), \tag{36}$$

and $\left(\Lambda\tilde{\Lambda}\right)^{-1}$ is obtained by replacing λ with $-\lambda$ in $\Lambda\tilde{\Lambda}$. By using Eq.(32) the Wigner function is

$$\begin{aligned} \langle \mathbf{0}| V^{-1}\Delta_3(\vec{q},\vec{p}) V |\mathbf{0}\rangle &= \frac{1}{\pi^3}\exp\left[-\frac{2}{3}(\cosh 2\lambda + 2\cosh\lambda)\sum_{i=1}^{3}|\alpha_i|^2\right] \\ &\times \exp\left\{-\frac{1}{3}(\sinh 2\lambda - 2\sinh\lambda)\sum_{i=1}^{3}\alpha_i^2\right. \\ &\left.-\frac{2}{3}\sum_{j>i=1}^{3}\left[(\cosh 2\lambda - \cosh\lambda)\alpha_i\alpha_j^* + (\sinh\lambda + \sinh 2\lambda)\alpha_i\alpha_j\right] + c.c\right\} \end{aligned} \tag{37}$$

For $n = 4$ case we have (see the Appendix)

$$\Lambda\tilde{\Lambda} = \begin{pmatrix} u' & w' & v' & w' \\ w' & u' & w' & v' \\ v' & w' & u' & w' \\ w' & v' & w' & u' \end{pmatrix}, \tag{38}$$

where $u' = \cosh^2\lambda, v' = \sinh^2\lambda, w' = -\sinh\lambda\cosh\lambda$. Then substituting Eq.(38) into Eq.(32) we obtain

$$\langle \mathbf{0}| V^{-1}\Delta_4(\vec{q},\vec{p}) V |\mathbf{0}\rangle = \frac{1}{\pi^4}\exp\left\{-2\cosh^2\lambda\left[\sum_{i=1}^{4}|\alpha_i|^2 + (M + M^*)\tanh^2\lambda + (R^* + R)\tanh\lambda\right]\right\}, \tag{39}$$

where $M = \alpha_1\alpha_3^* + \alpha_2\alpha_4^*$, $R = \alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_3\alpha_4$. This form differs evidently from the Wigner function of the direct-product of usual two two-mode squeezed states' Wigner functions (35). In addition, using Eq. (38) we can check Eqs.(25) and (26). Further, using Eq.(38) we have

$$N^{-1} = \frac{1}{2}\begin{pmatrix} 2 & \tanh\lambda & 0 & \tanh\lambda \\ \tanh\lambda & 2 & \tanh\lambda & 0 \\ 0 & \tanh\lambda & 2 & \tanh\lambda \\ \tanh\lambda & 0 & \tanh\lambda & 2 \end{pmatrix}, \; \det N = \cosh^2\lambda. \tag{40}$$

Then substituting Eqs.(40) and (A.4) into Eq.(18) yields the four-mode squeezed state,

$$V\left|0000\right\rangle = \mathrm{sech}\lambda \exp\left[-\frac{1}{2}\left(a_1^\dagger + a_3^\dagger\right)\left(a_2^\dagger + a_4^\dagger\right)\tanh\lambda\right]\left|0000\right\rangle, \tag{41}$$

from which one can see that the four-mode squeezed state is not the same as the direct product of two two-mode squeezed states in Eq.(1).

In sum, by virtue of the IWOP technique, we have introduced a kind of an n-mode squeezing operator $V \equiv \exp\left[\mathtt{i}\lambda\left(Q_1P_2 + Q_2P_3 + \cdots + Q_{n-1}P_n + Q_nP_1\right)\right]$, which engenders standard squeezing for the n-mode quadratures. We have derived V's normally ordered expansion and obtained the expression of n-mode squeezed vacuum states and evaluated its Wigner function with the aid of the Weyl ordering invariance under similar transformations.

Appendix: Derivation of Eq.(38)

For the completeness of this paper, here we derive analytically Eq.(38). Noticing, for the case of $n = 4$, $A^4 = I$, I is the 4×4 unit matrix, from the Cayley-Hamilton theorem we know that the expanding form of $\exp(-r\tilde{A})$ must be

$$\Lambda = \exp(-\lambda\tilde{A}) = c_0(\lambda)I + c_1(\lambda)\tilde{A} + c_2(\lambda)\tilde{A}^2 + c_3(\lambda)\tilde{A}^3. \tag{A.1}$$

To determine $c_j(\lambda)$, we take $\tilde{A}$ being $e^{\mathtt{i}(j/2)\pi}$ $(j = 0, 1, 2, 3)$ respectively, then we have

$$\begin{cases} \exp(-\lambda) = c_0(\lambda) + c_1(\lambda) + c_2(\lambda) + c_3(\lambda), \\ \exp(-\lambda e^{\mathtt{i}(1/2)\pi}) = c_0(\lambda) + c_1(\lambda)e^{\mathtt{i}(1/2)\pi} + c_2(\lambda)e^{\mathtt{i}\pi} + c_3(\lambda)e^{\mathtt{i}(3/2)\pi}, \\ \exp(-\lambda e^{\mathtt{i}\pi}) = c_0(\lambda) + c_1(\lambda)e^{\mathtt{i}\pi} + c_2(\lambda)e^{\mathtt{i}2\pi} + c_3(\lambda)e^{\mathtt{i}3\pi}, \\ \exp(-\lambda e^{\mathtt{i}(3/2)\pi}) = c_0(\lambda) + c_1(\lambda)e^{\mathtt{i}(3/2)\pi} + c_2(\lambda)e^{\mathtt{i}(6/2)\pi} + c_3(\lambda)e^{\mathtt{i}(9/2)\pi}. \end{cases} \tag{A.2}$$

Its solution is

$$\begin{cases} c_0(\lambda) = \frac{1}{2}\left(\cosh\lambda + \cos\lambda\right) \\ c_1(\lambda) = \frac{1}{2}\left(-\sinh\lambda - \sin\lambda\right) \\ c_2(\lambda) = \frac{1}{2}\left(\cosh\lambda - \cos\lambda\right) \\ c_3(\lambda) = \frac{1}{2}\left(-\sinh\lambda + \sin\lambda\right) \end{cases}. \tag{A.3}$$

It follows that

$$\Lambda = \begin{pmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{pmatrix}, \det\Lambda = 1, \tag{A.4}$$

and

$$\begin{aligned} \widetilde{\Lambda}\Lambda &= \left[c_0(\lambda)I + c_1(\lambda)A + c_2(\lambda)A^2 + c_3(\lambda)A^3\right]\cdot\left[c_0(\lambda)I + c_1(\lambda)\tilde{A} + c_2(\lambda)\tilde{A}^2 + c_3(\lambda)\tilde{A}^3\right] \\ &= \frac{1}{2}\begin{pmatrix} 2\cosh^2\lambda & -\sinh 2\lambda & 2\sinh^2\lambda & -\sinh 2\lambda \\ -\sinh 2\lambda & 2\cosh^2\lambda & -\sinh 2\lambda & 2\sinh^2\lambda \\ 2\sinh^2\lambda & -\sinh 2\lambda & 2\cosh^2\lambda & -\sinh 2\lambda \\ -\sinh 2\lambda & 2\sinh^2\lambda & -\sinh 2\lambda & 2\cosh^2\lambda \end{pmatrix}, \end{aligned} \tag{A.5}$$

this is just Eq.(38).

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References

[1] Bouwmeester D. et al., *The Physics of Quantum Information*, (Springer, Berlin) 2000.

[2] Nielsen M. A. and Chuang I. L., *Quantum Computation and Quantum Information* (Cambridge University Press) 2000.

[3] Buzek V., *J. Mod. Opt.*, **37** (1990) 303.

[4] Loudon R., Knight P. L., *J. Mod. Opt.*, **34** (1987) 709.

[5] Dodonov V. V., *J. Opt. B: Quantum Semiclass. Opt.*, **4** (2002) R1.

[6] Fan H.-y and Klauder J. R., *Phys. Rev. A* **49** (1994) 704.

[7] Fan H.-y and Fan Y., *Phys. Rev. A*, **54** (1996) 958.

[8] Fan H.-y, *Europhys. Lett.*, **23** (1993) 1.

[9] Fan H.-y, *Europhys. Lett.*, **17** (1992) 285; Fan H.-y, *Europhys. Lett.*, **19** (1992) 443.

[10] Fan H.-y, *J Opt B: Quantum Semiclass. Opt.*, **5** (2003) R147.

[11] Mandel L. and Wolf E., *Optical Coherence and Quantum Optics* (Cambridge University Press 1995) and references therein

[12] Einstein A., Poldolsky B. and Rosen N., *Phys. Rev.*, **47** (1935) 777.

[13] Wigner E. P., *Phys. Rev.*, **40** (1932) 749.

[14] O'Connell R. F. and Wigner E. P., *Phys. Lett. A*, **83** (1981) 145.

[15] Schleich W., *Quantum Optics* (New York: Wiley) 2001.

[16] Fan H.-y, *J. Phys. A*, **25** (1992) 3443; Fan H.-y, Fan Y., *Int. J. Mod. Phys. A*, **17** (2002) 701.

[17] Fan H.-y, *Mod. Phys. Lett. A*, **15** (2000) 2297.

[18] Fan H.-y, *Ann. Phys.*, **323** (2008) 500; **323** (2008) 1502.